

# Thermodynamics of the two-dimensional frustrated $J_1$ - $J_2$ Heisenberg ferromagnet in the collinear stripe regime: Susceptibility and correlation length

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We calculate the temperature dependence of the correlation length  $\xi$  and the uniform susceptibility  $\chi_0$  of the frustrated  $J_1$ - $J_2$  square-lattice Heisenberg ferromagnet in the collinear stripe phase using Green-function technique. The height  $\chi_{max}$  and the position  $T(\chi_{max})$  of the maximum in the  $\chi_0(T)$  curve exhibit a characteristic dependence on the frustration parameter  $J_2/|J_1|$ , which is well described by power laws,  $\chi_{max} = a(J_2 - J_2^c)^{-\nu}$  and  $T(\chi_{max}) = b(J_2 - J_2^c)$ , where  $J_2^c = 0.4$  and  $\nu$  is of the order of unity. The correlation length diverges at low temperatures as  $\xi \propto e^{A/T}$ , where  $A$  increases with growing  $J_2/|J_1|$ . We also compare our results with recent measurements on layered vanadium phosphates and find reasonable agreement.

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## I. INTRODUCTION

Frustrated square-lattice quantum magnets have been in the focus of active condensed-matter investigations in recent years. While for the study of ground state (GS) properties many alternative methods, such as exact diagonalization,<sup>1-4</sup> the Schwinger boson approach,<sup>5</sup> functional renormalization group method,<sup>6</sup> the tensor-product approach,<sup>7</sup> or the coupled-cluster method (CCM)<sup>4</sup> can be applied, there are much less reasonable theoretical approaches available to deal with thermodynamic properties of these systems. On the other hand, there are many recent experimental studies on quasi-two-dimensional frustrated square-lattice compounds, see, e.g., Refs. 8–16, where typically temperature dependent properties are reported, which should be compared with theoretical predictions. The quantum Monte Carlo technique is not applicable due to the sign problem for frustrated systems.<sup>17</sup> The high-temperature expansion approach<sup>18,19</sup> is limited to temperatures down to the order of the exchange coupling. Full exact diagonalization studies used in Refs. 20 and 21 for the  $J_1$ - $J_2$  square-lattice Heisenberg ferromagnet of  $N = 16$  and 20 sites exhibit strong finite-size effects at lower temperatures.<sup>19,22</sup> An appropriate method to describe quantum magnets in the whole temperature range is the Green-function technique.<sup>23-25</sup> A rotationally invariant second-order Green-function theory has been applied successfully to describe the thermodynamics of frustrated quantum magnets.<sup>22,26-30</sup>

Motivated by recent measurements of the correlation lengths for several frustrated layered Heisenberg square-lattice ferromagnets,<sup>15</sup> in the present paper we calculate the temperature dependence of the correlation length and

the uniform susceptibility of the spin-1/2  $J_1$ - $J_2$  model

$$H = J_1 \sum_{\langle i,j \rangle} \mathbf{S}_i \mathbf{S}_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} \mathbf{S}_i \mathbf{S}_j, \quad (1)$$

where  $\langle \dots \rangle$  denotes the nearest neighbor (NN) and  $\langle\langle \dots \rangle\rangle$  the next-nearest neighbor (NNN) bonds on a square lattice. We consider a ferromagnetic NN coupling  $J_1 < 0$  and a frustrating antiferromagnetic NNN coupling  $J_2 > 0$ . The GS of this model has been discussed in Refs. 2,4,5. At  $J_2 = J_2^c \approx 0.4|J_1|$  the ferromagnetic GS present at small  $J_2$  gives way for a GS phase with zero total magnetization. Although, the nature of this state for  $J_2 \gtrsim J_2^c$  is still under debate, the existence of antiferromagnetic collinear stripe GS long-range order (LRO) for  $J_2 \gtrsim 0.6|J_1|$  is not questioned.

Following our previous investigation of this model, see Ref. 22, we use the rotationally invariant second-order Green-function method (RGM) to calculate the thermodynamic properties. However, by contrast to Ref. 22, where the model was studied in the ferromagnetic regime, i.e.  $J_2 < J_2^c$ , here we focus on the antiferromagnetic collinear stripe GS regime  $J_2 \gtrsim 0.6|J_1|$ . This parameter regime corresponds to the experimental situation for layered vanadium phosphates.<sup>8,9,15</sup>

The paper is organized as follows. In Sec. II the RGM applied to the  $J_1$ - $J_2$  square lattice is illustrated. The results for the susceptibility and correlation length and the comparison to recent experimental data is presented in Sec. III. Finally, a short summary is given in Sec. IV.

## II. ROTATIONALLY INVARIANT GREEN-FUNCTION METHOD (RGM)

The RGM was introduced by Kondo and Yamaji.<sup>31</sup> The method was further developed and applied to Heisenberg magnets by several groups, see, e.g., Refs. 22,26–30,32–41.

To calculate the dynamical transverse spin susceptibility  $\chi_q^{+-}(\omega)$  we have to determine the two-time commutator Green function  $\langle\langle S_q^+; S_{-q}^- \rangle\rangle_\omega = -\chi_q^{+-}(\omega)$ . The equation of motion for  $\langle\langle S_q^+; S_{-q}^- \rangle\rangle_\omega$  up to second order reads  $\omega^2 \langle\langle S_q^+; S_{-q}^- \rangle\rangle_\omega = M_q + \langle\langle -\ddot{S}_q^+; S_{-q}^- \rangle\rangle_\omega$  with  $-\ddot{S}_q^+ = [[S_q^+, H], H]$  and the exact moment

$$M_q = -8J_1 C_{1,0} (1 - \gamma_q^{(1)}) - 8J_2 C_{1,1} (1 - \gamma_q^{(2)}), \quad (2)$$

where  $\gamma_q^{(1)} = (\cos q_x + \cos q_y)/2$  and  $\gamma_q^{(2)} = \cos q_x \cos q_y$ .  $C_{n,m}$  denotes the correlation functions. Supposing rotational symmetry, i.e.,  $\langle S_i^z \rangle = 0$ , they read  $C_{n,m} = C_{\mathbf{R}} = \langle S_{\mathbf{R}}^z S_0^z \rangle = 2\langle S_{\mathbf{R}}^z S_0^z \rangle$  with  $\mathbf{R} = n\mathbf{e}_x + m\mathbf{e}_y$ . Calculating the second derivative  $-\ddot{S}_q^+$ , an approximation as indicated in Refs. 22,26–41 is used which implies the decoupling scheme

$$S_i^+ S_j^+ S_k^- = \alpha_{i,k} \langle S_i^+ S_k^- \rangle S_j^+ + \alpha_{j,k} \langle S_j^+ S_k^- \rangle S_i^+, \quad (3)$$

where the quantities  $\alpha_{i,k}$  are vertex parameters introduced to improve the decoupling scheme. Since an  $\alpha_{i,k}$  is a function of the lattice vector  $\mathbf{R}_i - \mathbf{R}_k$  connecting the sites  $i$  and  $k$ , in what follows we use the same notation as for the correlation functions  $C_{n,m}$ , i.e. a vertex parameter  $\alpha_{n,m}$  belongs to the lattice vector  $\mathbf{R} = n\mathbf{e}_x + m\mathbf{e}_y$ . We obtain  $-\ddot{S}_q^+ = \omega_q^2 S_q^+$  and

$$\chi_q^{+-}(\omega) = -\langle\langle S_q^+; S_{-q}^- \rangle\rangle_\omega = \frac{M_q}{\omega_q^2 - \omega^2}, \quad (4)$$

with

$$\omega_q^2 = 2 \sum_{k,l(=1,2)} J_k J_l (1 - \gamma_q^{(k)}) \times [K_{k,l} + 8\alpha_{1,k-1} C_{1,k-1} (1 - \gamma_q^{(l)})], \quad (5)$$

where  $K_{1,1} = 1 + 2(2\alpha_{1,1} C_{1,1} + \alpha_{2,0} C_{2,0} - 5\alpha_{1,0} C_{1,0})$ ,  $K_{2,2} = 1 + 2(2\alpha_{2,0} C_{2,0} + \alpha_{2,2} C_{2,2} - 5\alpha_{1,1} C_{1,1})$ ,  $K_{1,2} = 4(\alpha_{1,2} C_{1,2} - \alpha_{1,0} C_{1,0})$ , and  $K_{2,1} = 4(\alpha_{1,0} C_{1,0} + \alpha_{1,2} C_{1,2} - 2\alpha_{1,1} C_{1,1})$ . The correlation functions  $C_{n,m}$  are calculated by the spectral theorem,<sup>23</sup>

$$C_q = \langle S_q^+ S_q^- \rangle = \frac{M_q}{2\omega_q} [1 + 2n(\omega_q)], \quad (6)$$

where  $n(\omega_q) = (e^{\omega_q/T} - 1)^{-1}$  is the Bose function. Magnetic LRO is reflected by a non-vanishing condensation term  $C$  (see, e.g., Refs. 27,32,34) according to

$C_{\mathbf{R}} = \frac{1}{N} \sum_{\mathbf{q} \neq \mathbf{Q}} C_{\mathbf{q}} e^{i\mathbf{q}\mathbf{R}} + C e^{i\mathbf{Q}\mathbf{R}}$  where  $\mathbf{Q}$  denotes the magnetic ordering vector. The magnetic order parameter (sublattice magnetization) is calculated by

$$m^2 = \frac{3}{2} \frac{1}{N} \sum_{\mathbf{R}} C_{\mathbf{R}} e^{-i\mathbf{Q}\mathbf{R}} = \frac{3}{2} C. \quad (7)$$

The corresponding static susceptibility is given by  $\chi_{\mathbf{Q}} = \frac{1}{2} \lim_{\mathbf{q} \rightarrow \mathbf{Q}} \chi_{\mathbf{q}}^{+-}(\omega = 0)$ . The uniform static susceptibility is  $\chi_0 = \frac{1}{2} \chi_{\mathbf{q}=0}^{+-}(\omega = 0)$ . The correlation length is obtained by an expansion of the static susceptibility around the magnetic ordering vector  $\mathbf{Q}$ ,  $\chi_{\mathbf{Q}+\mathbf{q}}^{+-} \approx \chi_{\mathbf{Q}}^{+-} (1 - \xi_x^2 q_x^2 - \xi_y^2 q_y^2)$ , see, e.g., Refs. 22,27,28,34,38. From the on-site correlator  $\langle S_i^- S_i^+ \rangle$  and the operator identity  $S_i^- S_i^+ = \frac{1}{2} + S_i^z$  we get the sum rule

$$C_0 = \frac{1}{N} \sum_{\mathbf{q}} C_{\mathbf{q}} = \frac{1}{2}. \quad (8)$$

Considering the collinear stripe phase we have two equivalent magnetic ordering vectors,  $\mathbf{Q}_1 = (0, \pi)$  and  $\mathbf{Q}_2 = (\pi, 0)$ . To preserve square-lattice symmetry we follow Ref. 27 and calculate the correlation functions as  $C_{n,m} = (C_{n,m}^{(1)} + C_{n,m}^{(2)})/2$  with  $C_{n,m}^{(i)} = \frac{1}{N} \sum_{\mathbf{q} \neq \mathbf{Q}_i} C_{\mathbf{q}} e^{i\mathbf{q}\mathbf{R}} + C e^{i\mathbf{Q}_i\mathbf{R}}$ . Note that  $C$  is the same for  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$ . Analogously, we consider  $\chi_{\mathbf{Q}+\mathbf{q}} = (\chi_{\mathbf{Q}_1+\mathbf{q}} + \chi_{\mathbf{Q}_2+\mathbf{q}})/2$  to get the correlation length by expansion of  $\chi_{\mathbf{Q}+\mathbf{q}}$  for small  $\mathbf{q}$  yielding

$$\xi^2 = \frac{2J_2 C_{1,1}}{4J_1 C_{1,0} + 8J_2 C_{1,1}} + \frac{A}{\Delta}, \quad (9)$$

where  $\Delta = 2J_1^2 K_{1,1} + 4J_2^2 K_{2,2} + 2J_1 J_2 (K_{1,2} + 2K_{2,1}) + 16(J_1 + 2J_2)(J_1 \alpha_{1,0} C_{1,0} + 2J_2 \alpha_{1,1} C_{1,1})$  and  $A = -J_2^2 (32\alpha_{1,1} C_{1,1} + K_{2,2}) - 4J_1 J_2 (3\alpha_{1,0} C_{1,0} + \alpha_{1,2} C_{1,2})$ .

In the GS, LRO may exist, that is,  $C \neq 0$ .  $C$  is determined by<sup>32,35,39</sup>  $\chi_{\mathbf{Q}_1}^{-1} = \chi_{\mathbf{Q}_2}^{-1} = 0$  with

$$\chi_{\mathbf{Q}_i} = -\frac{-4J_1 C_{1,0} - 8J_2 C_{1,1}}{\Delta}. \quad (10)$$

Next we have to discuss the choice of the vertex parameters  $\alpha_{n,m}$ . Obviously, there are five different  $\alpha_{n,m}$  in Eq. (5) which have to be determined together with the corresponding correlation functions  $C_{n,m}$ . In addition, at zero temperature the condensation term  $C$  (describing magnetic LRO) has to be considered. To determine these quantities we can use the Fourier transformation of Eq. (6) providing five equations for  $C_{n,m}$ . Moreover, at zero temperature we use  $\chi_{\mathbf{Q}_1}^{-1} = \chi_{\mathbf{Q}_2}^{-1} = 0$  to calculate  $C$ , see above. Finally, only one equation, namely the sum rule (8), is left to find the vertex parameters. Hence, we have to introduce further approximations. In the case of a ferromagnetic GS ( $J_2 < J_2^c$ ) all correlation functions behave quite similar and a reasonable approximation is to set  $\alpha_{n,m} = \alpha$ , see, e.g., Refs. 28,32 and 22. This simple approximation was also used in Ref. 31 applying

the RGM to antiferromagnets. However, in the antiferromagnetic regime the correlation functions carry different signs and setting  $\alpha_{n,m} = \alpha$  leads to poor results at low temperatures. A significant improvement of the RGM results for antiferromagnets can be achieved by introducing two independent vertex parameters.<sup>27,32,39</sup> This requires, however, an additional external input to get one more equation. To take into account the dominant character of  $J_2$  in the collinear stripe phase we set  $\alpha_{1,1} = \alpha_1$  and  $\alpha_{n,m} = \alpha_2$ ,  $(n,m) \neq (1,1)$ . Since the low-temperature properties of the model are related to excitations above the GS, a realistic description of the GS is necessary. Therefore, in the present paper we use, as an additional external input, the GS sublattice magnetization calculated by the CCM.<sup>4</sup> Thus, describing GS magnetic ordering properly, we may expect that the RGM provides also a reasonable description of the low-temperature properties of the model. This input yields the required additional equation to determine the two independent vertex parameters  $\alpha_1$  and  $\alpha_2$  at  $T = 0$ . As a result we can also calculate the uniform static susceptibility  $\chi_0$  at  $T = 0$ , cf. Fig. 1.

For finite temperatures we need a reasonable ansatz for the temperature dependence of the ratio  $\alpha_2/\alpha_1$  of the vertex parameters, see, e.g. Refs. 27,32,34 and 40. We have tested several ansatzes to get a proper description of thermodynamic quantities in the whole temperature range, see below. To solve the system of RGM equations we use Broyden's method,<sup>42</sup> which yields the solutions with a relative error of about  $10^{-8}$  on the average. The momentum integrals are done by Gaussian integration. To find the numerical solution of the equations for  $T > 0$ , we start at high temperatures and decrease  $T$  in small steps. Below a certain (low) temperature  $T_0(J_2)$  no solutions of the RGM equations (except at  $T = 0$ ) could be found, since the quantity  $\Delta(T, J_2)$  in Eqs. (9) and (10) becomes exponentially small which leads to numerical instabilities. As expected, at large temperatures the concrete choice of the ratio  $\frac{\alpha_2}{\alpha_1}(T)$  becomes irrelevant, and even the simple approximation  $\alpha_{n,m} = \alpha$  yields results for  $\chi_0(T)$  which coincide with the data from the high-temperature expansion. At low temperatures a reasonable ansatz for the ratio  $\alpha_2/\alpha_1$  should (i) provide numerical data down to sufficiently low temperatures and (ii) yield coincidence of  $\chi_0(T = 0)$  determined by using the CCM input and  $\lim_{T \rightarrow 0} \chi_0(T)$  calculated with the ansatz for  $\frac{\alpha_2}{\alpha_1}(T)$ . The simplest way is to fix the ratio  $\frac{\alpha_2}{\alpha_1}$  to its value at  $T = 0$ . Tracing the RGM solution to very low temperatures we find that the ansatz

$$\frac{\alpha_2(T)}{\alpha_1(T)} = 1 + \left( \frac{\alpha_2(0)}{\alpha_1(0)} - 1 \right) e^{-\gamma T} \quad (11)$$

with the tiny exponent  $\gamma = 0.005$  is more appropriate to get numerically stable solutions at low  $T$ , see also Fig. 1. In what follows we use this ansatz to solve the RGM equations with two vertex parameters.

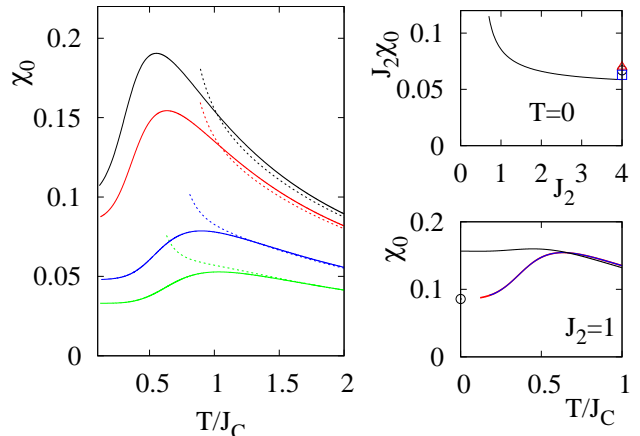


FIG. 1: Right: Temperature dependence of the uniform susceptibility  $\chi_0$  for  $J_2 = 0.9$  (black),  $J_2 = 1.0$  (red),  $J_2 = 1.5$  (blue), and  $J_2 = 2.0$  (green) calculated by RGM (solid lines). The temperature  $T$  is scaled by  $J_C = \sqrt{J_1^2 + J_2^2}$ . For comparison eighth order high-temperature expansion results (Refs. 18,19) are shown (dashed lines). Upper right: Ground-state values for  $J_2\chi_0$  vs.  $J_2$ . The susceptibility data for the square-lattice antiferromagnet (quantum Monte Carlo<sup>43</sup> - black circle, CCM<sup>45</sup> - red triangle, third-order spin-wave theory<sup>44</sup> - blue square) are also shown as data points at  $J_2 = 4$ . Lower right: Comparison of the temperature dependence of the uniform susceptibility  $\chi_0$  for  $J_2 = 1$  calculated by three different choices of the vertex parameters (red line - two vertex parameters with the ansatz (11), blue line - two vertex parameters with fixed ratio  $\frac{\alpha_2}{\alpha_1}(T) = \frac{\alpha_2}{\alpha_1}(0)$ , black line - one vertex parameter, black circle - RGM value for  $\chi_0$  at  $T = 0$ ). Note that the red and the blue lines practically coincide. Note further that the red line extends down to slightly lower temperatures.

### III. RESULTS AND DISCUSSION

In what follows we fix the ferromagnetic NN exchange to  $J_1 = -1$ . We focus on large values of  $J_2$ , where GS magnetic LRO is well established. Moreover, the experimental data for layered vanadium phosphates<sup>8,9,15</sup> correspond to this parameter regime. We follow Refs. 9 and 15 and use as a *characteristic energy scale*  $J_C = \sqrt{J_1^2 + J_2^2}$ .

In the left panel of Fig. 1 the temperature dependence of the uniform susceptibility  $\chi_0$  is shown. For comparison we also present the results of the eighth order high-temperature expansion,<sup>19</sup> which agree with the RGM data at large  $T$ . In the upper right panel of Fig. 1 the GS results for the susceptibility  $\chi_0(T = 0)$  are presented. Since for large  $J_2$  the  $J_1$ - $J_2$  model corresponds to a system of two inter-penetrating square-lattice antiferromagnets with coupling strength  $J_2$ , our RGM data for  $\chi_0(T = 0)$  can be compared with available GS results for  $\chi_0(T = 0)$  of the square-lattice antiferromagnet,<sup>43-45</sup> see data points at  $J_2 = 4$  in the upper right panel of Fig. 1.

For large  $J_2$  the dependence of  $J_2\chi_0(T=0)$  on  $J_2$  is weak down to  $J_2 \sim 1$ . A noticeable upturn of  $J_2\chi_0(T=0)$  for small  $J_2$  indicates the approach to the ferromagnetic phase. In the lower right panel of Fig. 1 we compare different choices of the vertex parameters to solve the RGM equations, see the discussion in Sec. II. Obviously, the use of only one vertex parameter by setting  $\alpha_{n,m} = \alpha$  leads to poor results for  $T \lesssim 0.6J_C$ .

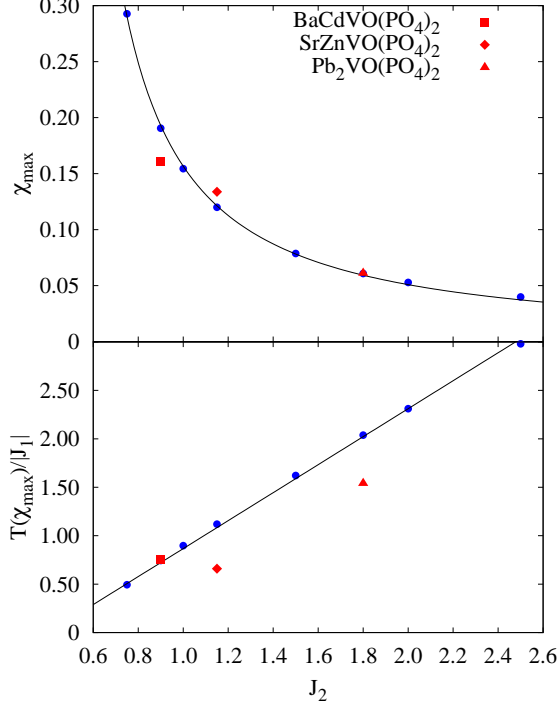


FIG. 2: Height and position of the maximum of the susceptibility  $\chi_0(T)$  in dependence on the frustration parameter  $J_2$ . The black filled circles are the RGM results, the lines correspond to Eqs. (12) and (13), and the colored symbols correspond to  $\text{BaCdVO}(\text{PO}_4)_2$  ( $J_2/|J_1| \approx 0.9$ )<sup>9</sup>,  $\text{SrZnVO}(\text{PO}_4)_2$  ( $J_2/|J_1| \approx 1.15$ )<sup>15</sup>, and  $\text{Pb}_2\text{VO}(\text{PO}_4)_2$  ( $J_2/|J_1| \approx 1.8$ )<sup>8</sup>.

For comparison with experiment, the height  $\chi_{max}$  and the position  $T(\chi_{max})$  of the maximum in the  $\chi_0(T)$  curve are interesting features. We have plotted these quantities as a function of  $J_2$  in Fig. 2. The RGM data points are well described by the relations

$$\chi_{max} = a(J_2 - J_2^c)^{-\nu}, \quad (12)$$

$$T(\chi_{max}) = b(J_2 - J_2^c) \quad (13)$$

with  $a = 0.0872$ ,  $\nu = 1.146$ ,  $b = 1.444$ , and  $J_2^c = 0.4$ . Using experimental data for the susceptibility for quasi-two-dimensional frustrated square-lattice magnets as well as the reported exchange constants  $J_1$  and  $J_2$  we can compare our theoretical data directly with experiment, see Fig. 2. Obviously, theory and experiment agree well, particularly for  $\chi_{max}$ . Hence, our equations (12) and

(13) can be used to estimate the  $J_2/|J_1|$  ratio from susceptibility measurements.

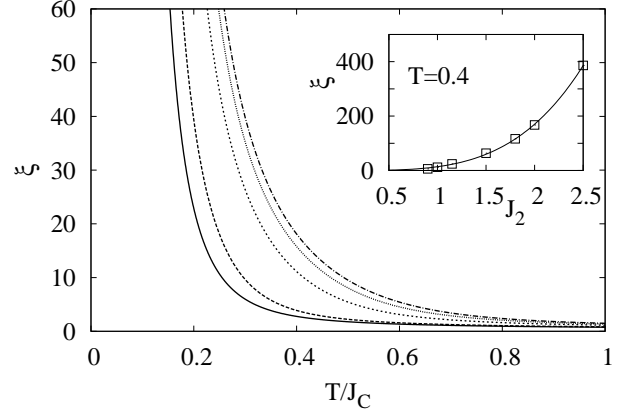


FIG. 3: Temperature dependence of the correlation length  $\xi$  for different values of the frustration parameter  $J_2=0.9, 1.0, 1.5, 2.0$  and  $2.5$  (from left to right). The temperature  $T$  is scaled by  $J_C = \sqrt{J_1^2 + J_2^2}$ . The inset shows the correlation length for  $T = 0.4$  in dependence on the frustration parameter  $J_2$ . The solid line represents a fit of the data points (open squares).

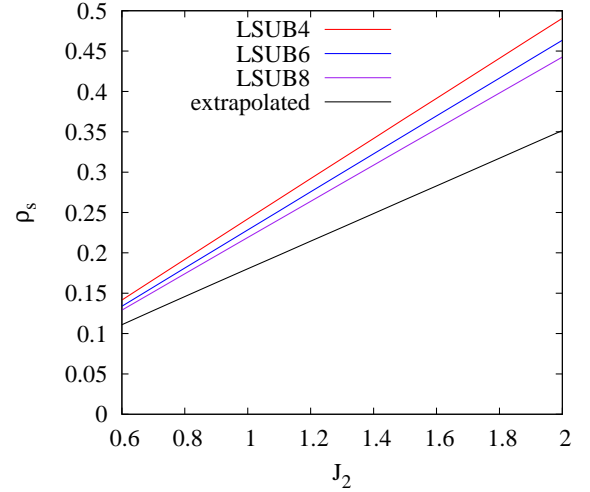


FIG. 4: Spin stiffness  $\rho_s$  as a function of  $J_2$  calculated by the CCM.

Next we discuss the correlation length  $\xi$ . Its temperature dependence is depicted in Fig. 3 for different values of the frustration parameter  $J_2$ . With increasing NNN exchange  $J_2$  the rapid increase in  $\xi$  is shifted to larger temperatures. As shown in the inset of Fig. 3 at a certain fixed temperature  $\xi$  decreases rapidly with decreasing  $J_2$ .

The exponential low-temperature divergence of  $\xi \propto e^{A/T}$  for two-dimensional Heisenberg magnets with NN

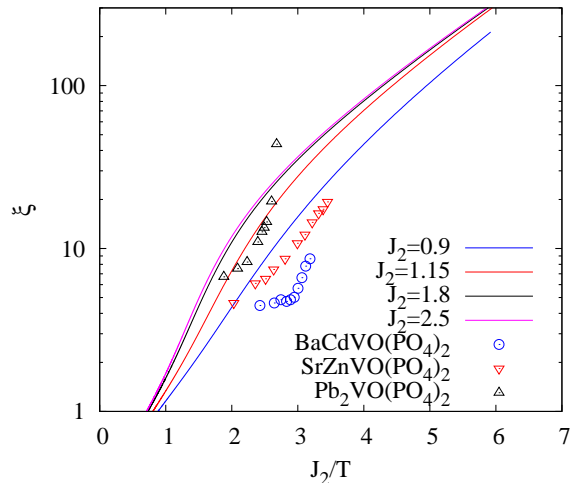


FIG. 5: Correlation length  $\xi$  (logarithmic scale) vs.  $J_2/T$  calculated using RGM (lines) for various values of the frustration parameter  $J_2$ . For comparison we also show experimental data<sup>15</sup> for  $\text{BaCdVO}(\text{PO}_4)_2$  ( $J_2/|J_1| \approx 0.9$ ),  $\text{SrZnVO}(\text{PO}_4)_2$  ( $J_2/|J_1| \approx 1.15$ ), and  $\text{Pb}_2\text{VO}(\text{PO}_4)_2$  ( $J_2/|J_1| \approx 1.8$ ).

interactions is determined by the spin stiffness  $\rho_s$ , i.e.  $A \propto \rho_s$ , see e.g. Refs. 22,47–51. As it has been recently reported,<sup>22</sup> for small  $J_2$ , where a ferromagnetic GS is present, the relation  $\xi \propto e^{a\rho_s/T}$  also holds if the NNN exchange  $J_2$  is included. In this case the stiffness was obtained from the RGM dispersion relation<sup>22</sup> as  $\rho_s^{FM} = -(J_1 + 2J_2)/4$ . For the antiferromagnetic collinear stripe GS phase present at large  $J_2$  one may expect that the stiffness also determines the exponential divergence at small  $T$ . However, the determination of  $\rho_s$  is more difficult. Here we use the CCM<sup>52–54</sup> to pro-

vide data for  $\rho_s$ . To calculate  $\rho_s$  within the CCM we follow strictly Refs. 53 and 54 and do not explain details of the calculation. The stiffness as a function of  $J_2$  is shown in Fig. 4 for various levels of CCM approximations, LSUB $n$ , as well as extrapolated data.<sup>55</sup> The obvious (almost) linear  $J_2$ -dependence of  $\rho_s$  is well described for the extrapolated CCM data by  $\rho_s \approx 0.175J_2$ . Hence, it seems to be reasonable to show the temperature dependence of the correlation length, in addition to Fig. 3, as a function  $\ln \xi(J_2/T)$ , see Fig. 5. First we notice that the experimental data reported in Ref. 15 agree reasonably well with our RGM results. Secondly, it is obvious that for large values of  $J_2 \gtrsim 1.5$  the  $\ln \xi(J_2/T)$  curves almost coincide. The small deviations can be attributed to a temperature dependent prefactor in front of the exponential term.<sup>22,47–51</sup> However, for  $J_2 = 1.15$  and  $J_2 = 0.9$  the theoretical as well as the experimental data show deviations from the behavior suggested by the stiffness data.

#### IV. SUMMARY

Using second-order Greens function technique we have calculated the uniform susceptibility  $\chi_0$  and the correlation length  $\xi$  of the frustrated  $J_1$ - $J_2$  square-lattice Heisenberg ferromagnet in the collinear antiferromagnetic regime present for large values of  $J_2/|J_1|$ . We have derived simple power laws for the height and the position of the maximum in the  $\chi_0(T)$  curve as functions of  $J_2$ . We have found that our theoretical data agree reasonably well with recent experiments on vanadium phosphates.

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